The Ladder of Time B: Time Rigidity Theorem—Thermodynamic Capping and the Logical Boundary of Causal Paradoxes

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Abstract

If time can be modified hierarchically in a physical system, it must face the dual constraints of operational cost and logical boundaries. This paper proves that time upgrades necessarily reach a fixed point after finitely many steps, and that this fixed point is unique. The core argument is based on a quantitative derivation of thermodynamic operational costs: each time upgrade involves the physical erasure of at least one bit of information, consuming at least the minimal work $k_B T \ln 2$ according to Landauer's principle; the total information budget of the system is bounded by the Bekenstein information-energy bound. From this we derive the maximum level $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$ and prove its uniqueness under fixed total energy and environmental temperature. Furthermore, as the level approaches the critical value N_c, causal self-referential operations lead to the emergence of closed causal chains, destroying the transitivity of the causal order. However, this logical critical point lies above the thermodynamic fixed point, making it physically inaccessible due to energy bankruptcy. Thermodynamic capping constitutes the actual insurmountable boundary for time upgrades, while the causal paradox cutoff serves as a logical safety redundancy. Together, the two constraints lock the unique optimal time level.

Keywords: Time hierarchy; Fixed point; Landauer's principle; Bekenstein bound; Causal paradox; Information budget; Partial order dimension

1 Introduction

The role of time in physics has undergone a profound transformation from an absolute background to a dynamical variable. Newtonian mechanics treated time as an externally given uniformly flowing parameter; general relativity showed that spacetime geometry is dynamically determined by matter distribution; quantum gravity research further reduces time to partial order relations among elementary events. The common thrust of these paradigm shifts is that time is not an a priori stage but an emergent property of the internal organization of physical processes.

A deeper question emerges from this: if the structure of time itself can be modified by the system, does such modification have an insurmountable endpoint? Each level of upgrade objectifies the lower level's temporal organization into operational objects of the higher level, but each step increases operational complexity. Starting from a discrete causal order, the system acquires layer by layer the ability to modify causal rules, meta-rules, and even the rules for modifying rules. If this hierarchical recursion were to continue indefinitely, it would inevitably hit the dual limits of physical resources and logical self-consistency.

In this paper, we proceed from two paths—operational physics and causal logic—to prove that there exists a unique optimal level for time upgrades. At this level, time already possesses irreversibility and causal closure, but has not yet fallen into paradoxes due to excessive self-reference. The core proof does not rely on heuristic analogies; instead, it is a quantitative derivation based on the first principles of thermodynamics: time upgrade, as a physical operation, incurs linearly accumulating energy costs and, under finite total energy, must inevitably cap; the capping position is uniquely determined by the system's total energy and the environmental temperature, thereby guaranteeing the uniqueness of the fixed point. The causal paradox cutoff is shown to lie above the thermodynamic capping, serving only as a logical safety redundancy rather than an actual boundary.

2 Physical Encoding of Time Structure and Formal Definition of the Objectification Operation

For the hierarchical transition of time to be subject to the dual constraints of thermodynamics and logic, the abstract causal order structure must first be translated into bit strings that a physical system can actually operate on, and the physical steps of the core operation called "objectification" must be rigorously defined. Without this translation, the "one bit" in Landauer's principle loses its referent, and the subsequent energy accounting would float above formal derivations. The task of this chapter is to establish this physical encoding mapping, clarify the sub-processes of the objectification operation, prove that the minimum information erasure is necessarily one bit, and ground the quantitative relationship between information cost and partial order complexity.

Consider the n-th level time structure described as a finite poset $(P_n, \leq_n)$, where P_n is the set of events, its cardinality denoted by $m_n = |P_n|$, and $\leq_n$ is the causal dependence partial order among events. To turn this mathematical structure into an entity that can be read, modified, and discarded within a physical system, we must construct an embedding map Φ_n that encodes $(P_n, \leq_n)$ into a finite-length string consisting of binary states. A natural and lossless scheme is to use the causal adjacency matrix representation. Define the matrix $A_n \in \{0,1\}^{m_n \times m_n}$ whose entries satisfy $A_n(i,j) = 1$ if and only if $i \leq_n j$, and 0 otherwise. Then $\Phi_n : (P_n, \leq_n) \longmapsto A_n$ gives a mapping from the abstract structure to a physical record of length $L_n = m_n^2$ bits. This encoding scheme does not require matrix storage to be physical storage; as long as we admit that any physical memory of the system can be realized by a set of bistable elements, the mapping converts the logical organization of the causal order into a thermodynamically measurable information object. To simplify the discussion, this paper does not introduce compression algorithms for the causal matrix, because any compression based on Kolmogorov complexity cannot alter the minimum thermodynamic dissipation corresponding to physical erasure—the Landauer lower bound depends only on the statistical mechanical cost of the erased bit, not on its higher-level semantic content.

The objectification operation, i.e., the complete action of upgrading from the n-th level to the $(n+1)$-th level, can be decomposed into the following four physical sub-processes:

1. **Readout**: The $(n+1)$-th level control unit acquires a representation of the causal matrix A_n from the n-th level storage through a reversible measurement or copying mechanism. If this step is completely reversible (e.g., via adiabatic quantum evolution or reversible logic gates), it in principle produces no net energy consumption. However, it is merely a preparatory operation; the true irreversible cost occurs in the subsequent disposal stage.

2. **Meta-encoding**: At the $(n+1)$-th level, an extended causal record is constructed. This record contains not only a copy of A_n, but must also be supplemented with a set of meta-rules R_n that specify which causal connections can be modified by higher-level operations, the permission levels for modification, and the logical constraints according to which modifications are made. The information amount of the meta-rules is denoted by $I_n^{(\text{meta})}$. In the simplest setting, R_n can be merely a "modification permission bit vector" whose length is of the same order as the number of causal pairs in P_n. Consequently, the total number of physical bits held by the $(n+1)$-th level is at least $m_n^2 + I_n^{(\text{meta})}$.

3. **Discarding of old rules**: The essence of a time upgrade is not a simple stacking of causal rules, but a replacement of old rules—there must be a set of old causal connections or meta-rules that are discarded, in order to manifest the "authority difference" between levels. This disposal action means that certain bits stored in the physical medium have their state forcibly changed from "old rule in effect" to "new rule in effect". If the medium had reached thermal equilibrium with the environment before rewriting, an irreversible rewriting process will lead to associated entropy production. Therefore, at least one bit's physical state must be erased to realize a non-trivial rule replacement. Concretely, suppose that some bit b in the n-th level records the permission status of a causal association. The meta-rules of the $(n+1)$-th level require this permission status to be flipped; then the original logical value of b must be irreversibly erased—either by coupling it to a heat bath and dissipating at least $k_B T \ln 2$ of heat, or by transferring its information to the environment and eventually discarding it. In thermodynamic terms, this is the direct action point of Landauer's principle.

4. **Verification**: The $(n+1)$-th level system confirms that the modified causal graph still maintains a strict partial order (acyclic, antisymmetric, transitive) and that no contradiction exists between the new and old rules. The verification process can be viewed as a logical constraint check and can be implemented via reversible computation, imposing no obligatory extra energy consumption; however, if a contradiction is detected and backtracking and correction are required, additional erasures may be triggered.

Thus, the minimum energy cost of a single objectification operation is rigorously anchored as one bit's erasure energy: any physical realization capable of bearing the operational semantics "old rule replaced by new rule" must at least flip the state of one binary storage unit, and consequently must dissipate at least $W_{\min} = k_B T \ln 2$ of heat into an environment at temperature T. Reversible computation strategies (Bennett, 1973) can defer the erasure to the end of the entire computation chain, but cannot eliminate the final disposal cost, because the information of the ultimately replaced old rules must be turned into heat; otherwise the system would accumulate infinitely many redundant causal paths, leading to behavioral indeterminacy and breaking the transitivity of the partial order. Hence, the minimum energy consumption per upgrade is unavoidable. This conclusion is independent of the specific implementation of the computing architecture and is a rigid corollary of the second law of thermodynamics applied to information processing (Landauer, 1961).

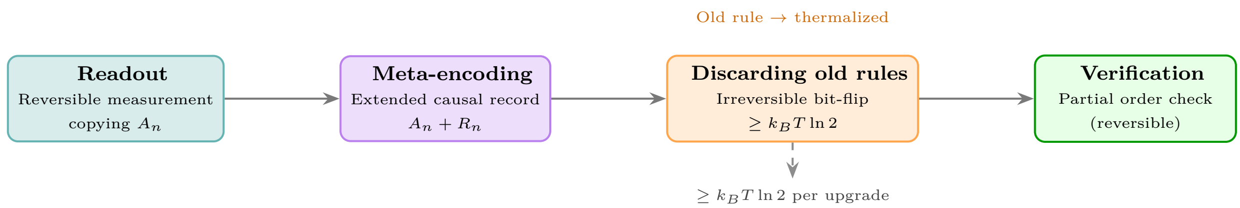

Figure 1: The four sub-processes constituting one objectification upgrade. Steps and are in principle reversible; Step (discarding old rules) is the irreversible node where Landauer's lower bound $k_B T \ln 2$ applies inevitably.

To establish a quantitative relationship between information cost and the complexity of the partial order structure, one must examine the information increment that must be handled at each upgrade. Let the number of new events introduced in the n-th upgrade be $\Delta m_n = m_{n+1} - m_n$; then the size of the causal matrix expands from m_n^2 to m_{n+1}^2, and the number of newly added bits is at least $\Delta m_n^2 = 2m_n \Delta m_n + (\Delta m_n)^2$. In addition, the meta-rules R_n occupy $I_n^{(\text{meta})}$ bits. If the upgrade only modifies rules without adding new events ($\Delta m_n = 0$), the information increment comes entirely from the meta-rules and the bit flips they induce. In any case, the number of bits that must be erased at each upgrade is at least 1, so we have the information consumption lower bound:

$$I_n \geq \max\{1,\ \Delta m_n^2 + I_n^{(\text{meta})}\} \text{ bits.}$$

This inequality strengthens the abstract "one bit" lower bound into a more refined estimate: in the case of substantially expanding the event set, the information cost may far exceed one bit. However, the thermodynamic capping theorem relies only on the most conservative lower bound "at least one bit erased per upgrade". Therefore, even by adopting this minimum, the derived maximum level $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$ is already the most generous upper limit among all possible realizations; actual physical systems will hit energy bankruptcy at a lower level due to higher information costs. This quantitative relationship does not weaken the uniqueness and universality of the theorem; on the contrary, it guarantees its robustness as an absolute upper bound.

Furthermore, from the perspective of partial order dimension, the irreducible order relations between the newly added meta-rules R_n and the original causal structure imply that the partial order dimension must increase by at least 1 (Dushnik & Miller, 1941; a formal proof of the dimension growth law can be found later and in the appendix). The addition of each independent order direction requires at least one extra bit for its representation at the encoding level. Therefore, the information cost lower bound is consistent with the dimension increment law, and both point to a common fact: a time upgrade is physically necessarily consumptive, and the amount of consumption is positively correlated with the scale of the level expansion.

The physical encoding and operational decomposition established in this chapter provide the necessary foundation for the subsequent accumulation of thermodynamic costs, the constraint of the Bekenstein information bound, and the proof of the uniqueness of the fixed point. It grounds Landauer's principle from a general prescription into a concrete energy consumption operator in the time upgrade scenario, elevating the assertion "each upgrade consumes at least $k_B T \ln 2$" from a heuristic to an operationally derived conclusion. At this point, the physical picture of time-level transitions is closed: the abstract causal order is transformed into a physical bit string via matrix encoding, each step of the objectification operation can be matched to concrete bit manipulations and heat flows, and the existence of a minimum energy consumption is locked onto the irreversible node of discarding old rules.

3 Time Upgrade as a Physical Operation

After the physical encoding mapping and the objectification operation flow have been established, the physical meaning of a time upgrade can be precisely formulated. This chapter strictly characterizes a time upgrade as a rewriting operation on the system's internal causal rules, and on this basis establishes a lower bound law for its energy cost.

Denote the n-th level time structure as the finite poset $(P_n, \leq_n)$, where P_n is a labelable set of events and $\leq_n$ is the causal dependence partial order among events. The essence of a time upgrade does not lie in fully reproducing all details of the lower-level causal structure, but in extracting and encoding the set of rules that generate that causal structure—namely the "meta-description". In accordance with the distinction made in the previous chapter, this paper adopts a weak objectification scheme: what the system constructs at the $(n+1)$-th level is not a pairwise relational copy of $(P_n, \leq_n)$ (strong objectification), but the meta-rules R_n capable of generating and modifying $(P_n, \leq_n)$. The information amount of the meta-rules is bounded by the minimum algorithmic description length needed to generate that causal order—namely its Kolmogorov complexity $K(P_n, \leq_n)$ (Solomonoff, 1964; Li & Vitányi, 2008)—which in most situations is far smaller than the explicit scale m_n^2 of the causal matrix.

Even under weak objectification, a time upgrade still must pay an irreducible physical price. The operational semantics of the upgrade demands that the new meta-rules R_{n+1} replace at least one valid causal generation instruction in the old meta-rules R_n. This replacement implies that a certain bit carrying the physical state of an old instruction is flipped and thermalized into the environment, triggering Landauer erasure. Concretely, let some bit b in the old meta-rules R_n record the permission status "event a causally precedes event b", while the new meta-rules R_{n+1} require this permission status to be negated. If the physical state of b had been stably stored and had reached thermal equilibrium with the environment before the upgrade, then irreversibly resetting it to the new logical value inevitably dissipates at least $k_B T \ln 2$ of heat into an environment at temperature T (Landauer, 1961). This lower bound is independent of the algorithmic complexity of the meta-rules: whether the erased bit carries a simple causal pair or a highly compressed rule fragment, its statistical mechanical cost depends only on the environmental temperature and the irreversibility of erasure, not on the bit's higher-level semantic content.

Consequently, the minimum energy cost of a single time upgrade is strictly locked by the environmental temperature as $W_{\min} = k_B T \ln 2$. This conclusion does not depend on the specific scale of the n-th level causal structure, nor on the information amount of the meta-rules—as long as the upgrade involves a discarding-type rule replacement of at least one bit, the Landauer lower bound applies. Reversible computation strategies can postpone erasure until the end of the entire upgrade chain (Bennett, 1973), but cannot eliminate the heat dissipation required by the final disposal, because the information of the replaced old rules must be removed from the system; otherwise the system would accumulate infinitely many valid causal paths, leading to behavioral indeterminacy and breaking the antisymmetry of the causal order.

After n upgrades, the cumulative minimum work of the system satisfies the strict lower bound inequality:

$$W_{\text{total}}(n) \geq n \cdot k_B T \ln 2.$$

The inequality rather than equality is used here for three reasons. First, the meta-rule complexity of later-level upgrades may grow, and the number of bits erased in a single upgrade in real systems is often larger than one—when the information amount of R_n significantly exceeds that of R_{n-1}, the disposal operation may involve the simultaneous erasure of multiple bits. Second, although the readout and verification sub-processes can be designed as reversible, it is difficult to completely eliminate parasitic dissipation in actual physical implementations. Third, maintaining the operation of the upgrade control logic itself also requires continuous energy expenditure. Therefore, $n \cdot k_B T \ln 2$ constitutes the absolute lower bound of cumulative energy consumption rather than an exact equality. This property is crucial for the robustness of the subsequent capping theorem: the $n_{\max}$ derived from it will serve as the theoretical upper limit of reachable levels for any system; the reachable level of any actual system must be less than or equal to this value, and the existence and uniqueness of the fixed point are not shaken by real energy consumption being higher than the lower bound.

This linear lower bound law is independent of the specific content of the upgrade; it depends only on the number of operations and the environmental temperature, forming a universal skeleton for the thermodynamic cost of time upgrades. It does not assume a particular way in which causal structure grows with levels, nor does it rely on a specific recurrence form of the partial order dimension, and is therefore applicable to any reasonably defined sequence of time levels—as long as each jump involves at least one bit of rule discarding, the energy cost accumulates with a linear lower bound.

Moreover, a direct corollary of the weak objectification scheme is that even if an upgrade only "fine-tunes" an old meta-rule—for example, merely modifying a single event-generation permission flag—the erasure cost is still no less than one bit. The strong objectification approach of copying the entire causal matrix would cause the information cost to grow as $O(m_n^2)$, but this paper proves that only the most conservative minimum is needed to derive the capping theorem. Hence, choosing weak objectification as the basis of the argument not only does not weaken the conclusion, but makes it more universal: even in the limiting case of the lowest information cost, thermodynamic bankruptcy remains unavoidable.

4 Information Budget and Energy Capping

The information processing capacity of a physical system is not unbounded; its total capacity is constrained by fundamental limits on energy and spatial scale. On the basis of having established time upgrade as a physical operation, this chapter introduces the Bekenstein bound as the upper limit of the information budget, and combines it with the thermodynamic energy cost equation to derive the level capping condition dominated by energy bankruptcy.

The total entropy that an isolated, localized physical system can accommodate has a universal upper bound jointly determined by its total energy and size. Bekenstein (1981) proved that for a weakly self-gravitating system of radius R and total energy E, its thermodynamic entropy satisfies

$$S \leq \frac{2\pi k_B E R}{\hbar c},$$

where k_B is the Boltzmann constant, $\hbar$ is the reduced Planck constant, and c is the speed of light. Measured in Shannon information units, defining information capacity $I = S/(k_B \ln 2)$ (in bits), the bound can be equivalently expressed as the total amount of information the system can store or process:

$$I_{\max} = \frac{2\pi E R}{\hbar c \ln 2}.$$

This bound indicates that the information capacity of any finite system has a hard upper limit. The present paper uses Shannon information rather than von Neumann entropy as the information measure because the physical encoding scheme established in the previous chapter uses classical bits as the basic unit, and the Landauer lower bound also takes classical bit erasure as the operational benchmark. If the underlying physical implementation is a quantum system, erasing a qubit has the same Landauer lower bound as erasing a classical bit (Reeb & Wolf, 2014); therefore, uniformly measuring in classical bits does not sacrifice universality and maintains the coherence of the information measure throughout the paper.

Time upgrade, as an information processing process, introduces new meta-rule information at each jump to extend or replace old causal records. Let the information consumption of the k-th upgrade be I_k (in bits); then the cumulative information consumption $\sum_{k=1}^{n} I_k$ cannot exceed the system's total information capacity $I_{\max}$:

$$\sum_{k=1}^{n} I_k \leq \frac{2\pi E R}{\hbar c \ln 2}.$$

The previous chapter has already established that each upgrade involves at least one bit of discarding-type erasure, so $I_k \geq 1$ bit. Under the most conservative estimate, the cumulative information consumption is at least n bits, growing linearly with the level. When n causes the cumulative information consumption to approach $I_{\max}$, the system is cut off in the information-theoretic sense: there is no longer enough entropy budget to accommodate the degrees of freedom of new meta-rules.

However, what the information budget constraint gives is merely a storage capacity limit, while the actual termination condition of time upgrades is triggered earlier by energy bankruptcy. Each upgrade must dissipate at least $k_B T \ln 2$ of heat into an environment at temperature T, with the cumulative energy consumption lower bound being

$W_{\text{total}}(n) \geq n \cdot k_B T \ln 2$. The total disposable energy E_{total} of the system must cover this cumulative work. When

$$n \cdot k_B T \ln 2 \geq E_{\text{total}}$$

holds, the system cannot extract enough energy from its internal degrees of freedom to carry out the erasure operation of the next upgrade. From this condition, the explicit expression for the maximum level is obtained:

$$n_{\max} = \left\lfloor \frac{E_{\text{total}}}{k_B T \ln 2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This $n_{\max}$ is directly locked by the energy account, irrespective of whether the information budget is exhausted—at the moment of energy bankruptcy, information storage capacity may still have redundancy, but the system is already unable to pay the energy cost required for discarding modifications of rules.

Here a crucial assumption about the system boundary is involved: the time upgrades considered in this paper occur within the reorganization of the system's internal degrees of freedom, without changing the system's total mass-energy E_{total} and its outer envelope radius R. The auxiliary degrees of freedom needed for each upgrade—i.e., the physical bits carrying the new meta-rules—all come from re-partitioning and re-encoding the existing degrees of freedom of the system, rather than introducing new matter or energy from the external environment. The "paper" and "ink" for constructing the meta-description are already taken from the system's own matter and energy budget; hence E_{total} remains constant throughout the upgrade sequence, and R similarly remains unchanged. This closed-upgrade assumption is self-consistent with the operational definition in the previous chapter that "the system constructs the meta-description internally": the objectification operation does not erect an architecture outside the system, but rearranges the organizational form of causal records within the system.

If this assumption is relaxed, allowing the system to absorb matter and energy from the environment to expand itself, then E_{total} and R will increase synchronously as the levels rise. In this case, $n_{\max}$ must be recalculated using the total energy of the expanded composite system, but the core conclusion of the theorem remains unchanged: for any finite system, there always exists a finite $n_{\max}$ that prevents further upgrades. The total energy of the composite system is still a finite value; the thermodynamic capping still exists, and the uniqueness of the fixed point is also unaffected.

It should also be pointed out that the conclusion that energy bankruptcy precedes information saturation relies on an order-of-magnitude comparison. In an actual physical system, suppose the system dissipates erasure heat in the form of electromagnetic radiation; then the number of levels satisfying $n_{\max}$ corresponds to the number of independent erasures that the system's total energy can support. In parallel, the information capacity given by the Bekenstein bound is usually much larger than the number of bits corresponding to that number of erasures—for example, for a system with total energy 1 joule, radius 0.1 meter, and environmental temperature 300 kelvin, $n_{\max}$ is on the order of 10^{18}, while $I_{\max}$ can exceed 10^{40} bits. Therefore, when energy is exhausted, information storage capacity is far from saturated. This quantitative relationship holds generally because the factor $2\pi R/(\hbar c \ln 2)$ in the Bekenstein bound is extremely large at macroscopic scales, while the factor $k_B T \ln 2$ in the Landauer cost is only on the order of 10^{-21} joules at room temperature. The energy account runs out before the information account due to the considerable accumulation of per-operation costs, which is precisely the physical origin of the statement that "thermodynamic capping constitutes the actual boundary, while the information budget constitutes a parallel but looser constraint".

In summary, the maximum level $n_{\max}$ of time upgrades is uniquely determined by the system's total energy and the environmental temperature through the Landauer lower bound. The Bekenstein information bound, as a parallel constraint, does not loosen the dominant position of energy, but provides a self-consistency check: if the cumulative information consumption corresponding to the $n_{\max}$ derived from energy exceeds $I_{\max}$, it indicates an inconsistency in the initial parameter settings—for example, the system cannot accommodate enough bits within the given R to realize $n_{\max}$ independent erasure operations, in which case $n_{\max}$ must be corrected downward. Within a reasonable parameter regime, this correction does not alter the existence and uniqueness of the fixed point, but merely results in a lower actual capping level. The joint action of the two constraints ensures that the conclusion of $n_{\max}$ as the level upper limit remains solid under the double test of thermodynamics and information theory.

5 Existence and Uniqueness of the Thermodynamic Fixed Point

On top of the energy accumulation lower bound and information budget constraint established in the previous chapter, this chapter presents a rigorous theorem on the thermodynamic capping of time levels and proves the uniqueness of that capping level.

Theorem 1 (Thermodynamic Capping Theorem)**.** *For any localized physical system with total energy E_{total} in an environment at temperature T, if its time upgrade sequence satisfies that each jump discards at least one bit of old-rule information, then there exists a finite maximum level $n_{\max}$, uniquely given by*

$$n_{\max} = \left\lfloor \frac{E_{total}}{k_B T \ln 2} \right\rfloor,$$

where k_B is the Boltzmann constant and $\lfloor \cdot \rfloor$ denotes the floor function. The system cannot execute the $(n_{\max}+1)$-th upgrade.

Proof. The analysis of physical encoding and objectification operations has established that the minimum energy consumption of a single upgrade is $W_{\min} = k_B T \ln 2$ (Landauer, 1961), and this lower bound still holds under the weak objectification scheme. After n upgrades, the cumulative energy consumption satisfies $W_{\text{total}}(n) \geq n \cdot k_B T \ln 2$. The total disposable energy E_{total} of the system constitutes the physical resource upper limit for the upgrade operations, so the condition $n \cdot k_B T \ln 2 \leq E_{\text{total}}$ must be met. From this we obtain $n \leq E_{\text{total}}/(k_B T \ln 2)$. Since n is a non-negative integer, the maximum level is $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$. The minimum energy consumption $k_B T \ln 2$ required for the $(n_{\max}+1)$-th upgrade would cause the cumulative demand to exceed E_{total}, violating the first law of thermodynamics, and thus cannot be executed. □ □

Theorem 2 (Uniqueness of the Fixed Point)**.** *Under fixed boundary conditions (E_{total}, T, R), the thermodynamic fixed point of time upgrades is unique.*

Proof. With E_{total} and T fixed, the Landauer lower bound gives a unique $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$. Consider an arbitrary level n, and distinguish three cases:

(1) If $n > n_{\max}$, then the cumulative energy consumption required to execute n upgrades is at least $n \cdot k_B T \ln 2 > E_{\text{total}}$, and the system's energy is insufficient to support it; this level is physically unreachable under the given boundary conditions.

(2) If $n < n_{\max}$, then $n \cdot k_B T \ln 2 \leq (n_{\max} - 1) \cdot k_B T \ln 2 < E_{\text{total}}$, and after completing n upgrades the system still has at least $k_B T \ln 2$ of residual disposable energy. This residual is sufficient to pay the energy cost of one more discarding-type rule replacement, so the system can continue upgrading to at least the $(n+1)$-th level. Hence, any level with $n < n_{\max}$ does not possess the closure property of an upgrade endpoint—it cannot become the terminal state of time-level recursion.

(3) $n = n_{\max}$ is the only level that simultaneously satisfies the following two conditions: first, its cumulative energy consumption lower bound does not exceed E_{total}, making it physically reachable; second, if one more step is taken, the required cumulative energy consumption would cross the E_{total} boundary, forcing the system to stop. This logical position of "reachable but the next step is unreachable" is precisely the defining characteristic of a thermodynamic fixed point.

The above boundary property of "exactly unable to continue" is uniquely determined by E_{total} and T through the expression for $n_{\max}$, independent of the concrete content of the causal structure or the growth pattern of meta-rule complexity. $n_{\max}$ is the critical index at which the energy account is exhausted.

The Bekenstein information bound (Bekenstein, 1981) serves here as a self-consistency check, not as a prerequisite for the uniqueness of the fixed point. The cumulative information consumption lower bound derived from $n_{\max}$ is $n_{\max}$ bits. The total information capacity given by the Bekenstein bound is $I_{\max} = 2\pi E_{\text{total}} R/(\hbar c \ln 2)$. If $n_{\max} \leq I_{\max}$, then energy capping is compatible with the information budget, and $n_{\max}$ is the effective fixed point. In the rare case where $n_{\max} > I_{\max}$ occurs, it indicates that the initially set E_{total} or R is insufficient to physically accommodate the storage resources required for $n_{\max}$ independent erasure operations; in this case one must take the information bound as the tight constraint and use the corrected value $n'_{\max} = \lfloor I_{\max} \rfloor$ as the actual capping level. This correction does not change the existence and uniqueness of the fixed point: the corrected $n'_{\max}$ is likewise the unique maximum level under the fixed boundary conditions, and the logical structure of the original theorem is completely preserved. In the macroscopic physical parameter regime, $I_{\max}$ usually far exceeds $n_{\max}$, and energy bankruptcy precedes information saturation; the correction of $n_{\max}$ by the information bound is minimal or zero, but in any case, a unique fixed point always exists. □

In summary, the thermodynamic fixed point $n_{\max}$ is uniquely locked by the system's total energy and the environmental temperature through the Landauer lower bound. Its existence stems from the rigid restriction of finite energy on the number of executable operations, and its uniqueness arises from the exclusivity of the critical condition "exactly unable to pay the energy cost of the next step": on the non-negative integer axis, there is one and only one integer n satisfying $n \cdot k_B T \ln 2 \leq E_{\text{total}}$ and $(n+1) \cdot k_B T \ln 2 > E_{\text{total}}$, namely $n_{\max}$.

Corollary 1. *Time upgrade is not an infinitely recursive process. Any energy-limited localized physical system must, in the thermodynamic sense, terminate the upgrade after finitely many steps and cannot complete an infinite hierarchy of jumps. This conclusion does not stem from speculations about computational complexity or logical paradoxes, but* *is a direct corollary of the first law of thermodynamics and Landauer's principle in the context of information erasure.*

6 Causal Paradox Cutoff as Logical Redundancy

Thermodynamic capping establishes an insurmountable boundary for time upgrades in the energy dimension. However, independently of the energy account, causal logic itself also sets an intrinsic upper limit for level jumps. When the level climbs to the point where the system can take its own causal order as the object of modification, the logical structure of self-referential operations will force the emergence of closed causal chains, thereby destroying the transitivity of the partial order. Starting from the growth law of partial order dimension, this chapter rigorously demonstrates the existence condition of this logical critical point, and proves that this critical point always lies above the thermodynamic bankruptcy point, making it physically inaccessible because energy is exhausted before logic.

Express the causal order of level n as a finite poset $(P_n, \leq_n)$. To quantify the number of independent order directions in its causal structure, we introduce the Dushnik-Miller dimension $\dim(P_n)$, defined as the minimum number of total orders whose intersection equals the partial order $\leq_n$ (Dushnik & Miller, 1941). Equivalently, $\dim(P_n)$ equals the minimum dimension d such that P_n can be embedded into the direct product of d total orders. A higher dimension means more mutually irreducible independent ordering directions in the causal structure.

The objectification operation of time upgrade requires that the meta-description of the n-th level causal order be constructed at the $(n + 1)$-th level. From the perspective of partial order structure, this operation produces at least the following effect: at the $(n+1)$-th level, it must be possible to distinguish "elements belonging to P_n" from "meta-descriptors about P_n". This distinction cannot be expressed using the independent order directions already existing inside P_n—because within the native partial order framework of P_n, meta-descriptors simply do not exist, let alone any partial order relations between them and the original events. Hence, the objectification operation necessarily requires that at least one new total order direction be introduced into the partial order structure of the $(n+1)$-th level, a direction that is not collinear with any of the original total orders making up $\dim(P_n)$. Otherwise, meta-descriptors and original events would be indistinguishable in terms of partial order relations, and the objectification would semantically fail.

We can rigorize the above inference using the embedding definition of Dushnik-Miller dimension. Suppose P_n can be embedded into the direct product of $d = \dim(P_n)$ total orders. If P_{n+1} could also be embedded into the direct product of only d total orders, then all meta-descriptors and their partial order relations with elements of P_n would have to be linearly composed from the existing d order directions. But this would mean that meta-descriptors are equivalent, in the sense of partial order, to certain linear combinations of events in P_n, thereby losing the semantic independence of "meta-level representation"—the meta-descriptor cannot be separated from what it describes. This contradicts the definition of the objectification operation. Therefore, the required embedding dimension for P_{n+1} is at least $d+1$, i.e.,

$$\dim(P_{n+1}) \geq \dim(P_n) + 1.$$

This inequality does not rely on constructive assertions or analogies; it is a direct corollary of the semantics of the objectification operation on the partial order dimension: the type distinction between coder and coded must occupy at least one independent order direction.

The dimension growth law harbors a profound consequence: as levels rise, the number of independent order directions in the causal structure accumulates at an at least linear rate. However, level n itself provides only n level indices. When the partial order dimension exceeds the level index, the problem of under-representation inevitably emerges.

Concretely, a system at level n possesses at most n "explicitly controllable order directions". A controllable order direction is a partial order dimension that the system can explicitly name, distinguish, and manipulate within the meta-description framework of level n. Each level rise introduces one new independent order direction via the objectification operation, so ideally the level index and the number of controllable order directions correspond one-to-one. But when the growth rate of the partial order dimension of the causal structure itself exceeds the level rise rate, we have $\dim(P_n) > n$. At this point, there exists at least one independent order direction in the system that cannot be represented by the level index—it exists but cannot be named by the meta-description of the n-th level.

During an upward objectification jump, the $(n+1)$-th level must complete the representation of all order directions of P_n. But when $\dim(P_n) > n$, the coder, whose capability is limited to $n+1$ order directions (the $(n+1)$-th level has only one more newly created order direction than the n-th level), faces a coded object P_n that possesses more than n independent order directions, resulting in a representation deficit. This under-representation forces the coder to "fold" at least one unrepresented order direction into its own representation framework—i.e., use one of its own existing order directions to simultaneously represent two independent order directions of the coded. In the partial order structure, this means that two originally independent partial order relations are forcibly identified. If there exists a non-trivial dependence relation between these two independent order directions, the identification operation will directly generate a closed causal chain of the form $a \to T_n(a) \to \cdots \to a$, where T_n is the action of the self-referential map on P_n. The emergence of a closed causal chain means that in $\leq_n$ there exist both $a \leq_n b$ and $b \leq_n a$, directly violating the antisymmetry of the partial order, causing a global collapse of the transitivity of the causal order.

Thus, the logical condition for causal degradation is established. Define the causal self-referential operator S_n, whose scope $\mathrm{Scope}(S_n) \subseteq P_{n+1}$ is the set of elements that can modify the causal order of P_n, and whose causal presupposition set $E_{\mathrm{pre}} \subseteq P_n$ is the minimal causal condition required for the execution of S_n. The formal condition for a closed causal chain is: there exists a map T_n such that $T_n(E_{\mathrm{pre}}) \subseteq \mathrm{Scope}(S_n)$, so that S_n modifies the very presuppositions of its own execution, locking in a loop. The critical level N_c is defined as the smallest n such that $\dim(P_n) > n$. Above N_c, under-representation inevitably triggers the folding operation, and no self-referential map can avoid circular dependence; the system enters the causal degradation phase.

Next, we argue the hierarchical relationship between the causal degradation phase and thermodynamic capping. Let the partial order dimension of the n-th level be $d_n =$

$\dim(P_n)$. From the dimension growth law, we have $d_n \geq d_1 + (n-1)$. Taking $d_1 \geq 1$—the initial level at least possesses a one-dimensional partial order—we obtain $d_n \geq n$ for all n. The condition for $d_n > n$ to occur is $d_n \geq n+1$, i.e., there exist at least $n+1$ independent order directions in the system.

Each independent order direction requires at least one bit in the meta-description to characterize its preference orientation. Representing d_n independent order directions requires at least d_n bits of meta-rule information. Under the physical encoding scheme, adding the representation of one independent order direction means introducing at least one new bit into the meta-rule set, and the discarding-type modification of that bit still requires paying the Landauer erasure cost. Let the maximum number of independent order directions that the system's total energy E_{total} can support be $d_{\max}$. Considering that the representation of each additional independent order direction corresponds to at least one bit of discarding-type rule replacement, we have $d_{\max} \cdot (k_B T \ln 2) \leq E_{\text{total}}$, i.e., $d_{\max} = n_{\max}$. By the thermodynamic capping theorem, $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$ is the maximum reachable level of the system.

Therefore, to satisfy $d_n > n$, i.e., $d_n \geq n+1$, we must have $n+1 > n_{\max}$ (otherwise the energy cost of the required $n+1$ independent order directions would already exceed E_{total}). From this we obtain $N_c \geq n_{\max}+1$. The logical critical point lies strictly above the thermodynamic bankruptcy point, with at least one full level separating them. Before the physical system has accumulated enough independent order directions to trigger under-representation, its energy account is already exhausted.

This conclusion has a clear operational meaning: the causal paradox cutoff does not constitute an actual boundary for time upgrades, but serves as a logical safety redundancy at a higher level. Even if a system could, in some hypothetical way, bypass the Landauer lower bound—for instance, via some mechanism not yet excluded by physics that avoids erasure energy consumption—the structure of causal logic itself would still block infinite upgrades at N_c. Thermodynamic capping stands in front, followed by the logical trap; the dual constraints ensure that the time-level jumps of any finite physical system must inevitably converge to a unique finite fixed point.

7 Conclusion

If time structure can be hierarchically modified within a physical system, this very capacity for modification is not an infinitely recursive freedom. Starting from two independent paths—operational physics and causal logic—this paper has proved that time upgrades must inevitably converge after finitely many steps to a unique fixed point, which is rigidly locked by thermodynamic capping, with the causal paradox cutoff serving as logical safety redundancy.

On the operational physics path, a time upgrade is established as an information erasure process involving the discarding of old rules. Through the physical encoding of the causal order and the decomposition of the objectification operation into sub-processes, the minimum energy cost of a single upgrade is rigorously anchored to the Landauer lower bound $k_B T \ln 2$, and the cumulative energy consumption satisfies the lower bound inequality $W_{\text{total}}(n) \geq n \cdot k_B T \ln 2$. Combined with the finiteness of the system's total energy E_{total}, we derive the maximum level $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$. This capping level is the unique fixed point satisfying “physically reachable but the next step is unreachable”—any level lower than it lacks the closure property of an upgrade endpoint, and any level higher than it is thermodynamically unexecutable. The Bekenstein information bound, as a parallel constraint, provides a self-consistency check, confirming that energy bankruptcy precedes information saturation in ordinary physical parameter regimes, thereby establishing thermodynamic capping as the actual insurmountable rigid boundary.

On the causal logic path, the objectification operation is proved to necessarily introduce new independent order directions, leading to the dimension growth law $\dim(P_{n+1}) \geq \dim(P_n)+1$. When the partial order dimension exceeds the level index, under-representation forces self-referential operations to fold independent order directions into their own representation framework, producing closed causal chains and destroying the transitivity of the causal order. This critical level N_c, through a rigorous inequality derivation, satisfies $N_c \geq n_{\max}+1$, invariably lying above the thermodynamic bankruptcy point. This means that the physical system's energy account is already exhausted before it has accumulated enough independent order directions to trigger causal degradation. The causal paradox cutoff thus does not constitute an actual boundary, but is a deeper logical insurance.

The two paths form a nested structure: thermodynamic capping is the first insurmountable barrier, and the causal paradox cutoff is a second logical defense that stands guard even under extreme hypothetical scenarios. Together, they lock the unique optimal level of time upgrades at $n_{\max}$—a level at which the system has completed $n_{\max}$ rewritings of causal rules, where time possesses irreversibility and causal closure, yet has not fallen into the self-referential degradation phase due to partial order dimension overflow.

The existence of this fixed point is not a heuristic guess, nor a one-sided inference from computational complexity or logical paradoxes. It is rooted in a direct derivation of the first law of thermodynamics and Landauer's principle in the context of information erasure, and simultaneously accepts the cross-validation of the partial order dimension growth law and the energy-information equivalence relation. The universality of the theorem covers situations ranging from microscopic quantum systems to macroscopic cosmological scales: as long as the system's energy is finite and the objectification operation involves at least one bit of rule discarding, convergence is inevitable, and the fixed point uniquely exists. The endpoint of time-level jumps is determined by the rigidity of physical laws, not by the choice of logical constructions.

Remark

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Appendix A: Rigorous Formulation of Landauer's Principle and the Bekenstein Bound, Physical Dimensions, and Applicability in Extreme Situations

This appendix aims to provide complete formal statements for the two core physical principles on which the main text relies—Landauer's erasure principle and the Bekenstein information-energy bound—and to argue that they still constitute effective lower bounds under extreme physical conditions, thereby ensuring the universality of the thermodynamic capping theorem.

A.1 Landauer's Principle

Landauer (1961) proved that any logically irreversible operation that loses one bit of information in an environment at temperature T is necessarily accompanied by a heat dissipation of at least

$$W_{\min} = k_B T \ln 2,$$

where k_B is the Boltzmann constant. The derivation of this lower bound relies only on the compression of phase-space volume before and after information erasure and the second law of thermodynamics, and is independent of the specific computing architecture—whether electronic logic gates, biomolecular switches, or quantum measurement devices. Reversible computation strategies (Bennett, 1973) permit delaying erasure, so that part of a computation chain can proceed without energy consumption, but any step that eventually discards information and resets a storage unit to a standard state must, by its physical nature, still pay this heat price. The Landauer limit is therefore incompressible in the operational sense: it does not represent an engineering approximate lower bound, but a rigid boundary drawn by statistical mechanics for logical operations.

Time upgrade, as a physical operation, necessarily involves the erasure of records of the replaced rules during the process of objectifying the old causal order and constructing callable rules; even if this erasure corresponds to the discarding of merely one bit of information, it dissipates at least $k_B T \ln 2$ of heat. Based on this, the main text establishes the minimum energy cost of a single upgrade and derives the lower bound inequality for cumulative energy consumption.

A.2 Bekenstein Bound and Information Budget

The total amount of information that an isolated, localized physical system can accommodate is not arbitrarily large. Bekenstein (1981) proposed a universal upper bound: for a weakly self-gravitating system of radius R and total energy E, its thermodynamic entropy satisfies

$$S \leq \frac{2\pi k_B E R}{\hbar c},$$

where $\hbar$ is the reduced Planck constant and c is the speed of light. Measured in Shannon information units, defining the information quantity $I \equiv S/(k_B \ln 2)$ (in bits), the inequality can be rewritten as

$$I_{\max}^{\text{Bek}} = \frac{2\pi E R}{\hbar c \ln 2}.$$

The total information budget that the system can acquire or store is uniquely locked by the total energy and spatial scale.

The original derivation of the Bekenstein bound assumes the system is weakly self-gravitating and approximately spherically symmetric. In the strong-gravity regime, when R approaches the system's Schwarzschild radius, the bound degenerates into the holographic bound ('t Hooft, 1993; Susskind, 1995), i.e., information capacity scales with the boundary area rather than the volume. The holographic bound has the form $S \leq A/(4l_P^2)$, where $A = 4\pi R^2$ is the area of the system's outer envelope and $l_P = \sqrt{\hbar G/c^3}$ is the Planck length. Measured in bits, this corresponds to

$$I_{\max}^{\text{holo}} = \frac{A}{4l_P^2 \ln 2} = \frac{\pi R^2}{l_P^2 \ln 2}.$$

In general physical situations, the actual information capacity of the system is constrained by the stricter of the two bounds: $I_{\max} = \min(I_{\max}^{\text{Bek}}, I_{\max}^{\text{holo}})$. Comparing the Bekenstein bound and the holographic bound: the Bekenstein bound is proportional to $E \cdot R$, while the holographic bound is proportional to R^2. When the total energy of the system satisfies $E > \frac{\hbar c}{2R} \cdot (R/l_P)^2$, the holographic bound tightens before the Bekenstein bound, and the information budget is determined by the boundary area rather than the bulk energy. For macroscopic systems, this crossover condition is easily satisfied, so the holographic bound constitutes a tighter constraint in most practical situations.

The impact of this comparison on the thermodynamic capping theorem is direct. If the holographic bound is taken as the upper limit of the information budget, the maximum number of levels the system can support must simultaneously satisfy the energy constraint and the holographic constraint:

$$n_{\max}^{\text{actual}} \leq \min\left(\left\lfloor \frac{E_{\text{total}}}{k_B T \ln 2} \right\rfloor, \left\lfloor \frac{\pi R^2}{l_P^2 \ln 2} \right\rfloor\right),$$

taking the stricter of the two. The main text uses the Bekenstein bound to derive $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$ because its form directly couples with energy, making it convenient to establish an explicit analytic relationship between the level index and the energy account. In strong-gravity or large-scale situations where the holographic bound tightens before the Bekenstein bound, the actual capping level will be further lowered, but the existence and uniqueness of the fixed point remain unchanged—the corrected $n_{\max}$ is still the unique maximum level under the fixed boundary conditions, only with a smaller numerical value. The holographic correction does not loosen the logical skeleton of the theorem; on the contrary, it makes the constraint of thermodynamic capping even more robust.

A.3 Landauer Lower Bound in the Quantum Regime

The quantum version of Landauer's principle (Reeb & Wolf, 2014) confirms that erasing a qubit has the same energy lower bound $k_B T \ln 2$ as erasing a classical bit; one cannot circumvent this fundamental cost by exploiting quantum entanglement or measurement back-action. Therefore, even if information is stored and represented in the form of quantum superpositions during the time upgrade process, the energy accounting law still strictly holds and is not loosened by the quantization of the underlying logic.

Appendix B: Formal Supplements on Causal Paradox Cutoff and Partial Order Dimension

This appendix provides the formal definitions and main derivations of the partial order dimension and self-referential causal paradox used in the main text, supplementing the more condensed arguments in the main text.

B.1 Partial Order Dimension (Dushnik-Miller Dimension)

Let $(P, \leq)$ be a finite poset. Its dimension $\dim(P)$ is defined as the minimum number of total orders whose intersection equals $\leq$ (Dushnik & Miller, 1941). Equivalently, $\dim(P)$ equals the smallest integer d such that P can be embedded into the direct product of d total orders. The dimension quantifies the number of independent ordering directions (i.e., the number of mutually irreducible types of causal arrows in the causal order) in the partial order structure.

Basic properties:

1. If Q is a sub-poset of P, then $\dim(Q) \leq \dim(P)$.
2. The dimension of a totally ordered set is 1; the dimension of a finite Boolean lattice equals its number of atoms.
3. $\dim(P \times \mathbf{2}) = \dim(P) + 1$, where $\mathbf{2}$ is the two-element chain $\{0 < 1\}$. This property precisely characterizes the dimension increment when a new total order direction is introduced.

B.2 Objectification Operation and Dimension Growth Law

The objectification operation of time upgrade requires constructing the meta-description of the n-th level causal order $(P_n, \leq_n)$ at the $(n+1)$-th level. From the perspective of partial order structure, this means constructing a new poset P_{n+1} that at least contains P_n as a sub-poset and additionally contains a set of meta-descriptors $M \subseteq P_{n+1} \setminus P_n$ used to characterize and encode P_n. There exist partial order relations between the meta-descriptors and the original events in P_n, and these relations cannot be linearly expressed by the independent order directions already present inside P_n—otherwise the meta-descriptors and the original events would be indistinguishable in the sense of partial order, and the objectification would fail semantically.

Below we give a rigorous proof of the dimension growth law $\dim(P_{n+1}) \geq \dim(P_n)+1$. We proceed by contradiction. Assume $\dim(P_{n+1}) = \dim(P_n) = d$. By the embedding definition of Dushnik-Miller dimension, there exists an order-preserving injection $f : P_{n+1} \to \mathbb{R}^d$, where $\mathbb{R}^d$ is equipped with the direct product partial order of d total orders. The map f expresses all partial order relations of P_{n+1} as linear combinations of the d independent order directions.

Consider a meta-descriptor $m \in M$. By the definition of the objectification operation, there exists an event $x \in P_n$ such that there is a partial order relation between m and x (for instance, m makes a judgment about the causal property of x in the meta-level description), and this relation cannot be expressed by the d independent order directions inside P_n. However, f maps m to a vector $f(m)$ in $\mathbb{R}^d$, and $f(x)$ is also a vector in $\mathbb{R}^d$; the partial order relation between them in $\mathbb{R}^d$ is entirely determined by comparisons in the d component directions. This means the meta-level partial order relation between m and x has been linearly expressed by the original d directions, contradicting the irreducibility assumption.

This contradiction shows that the original assumption $\dim(P_{n+1}) = d$ is false. Therefore, the embedding dimension of P_{n+1} is at least $d + 1$, i.e., $\dim(P_{n+1}) \geq \dim(P_n) + 1$. $\square$

B.3 Self-Referential Closed Causal Chains and Critical Level

Define the causal self-referential operator S_n: its scope $\mathrm{Scope}(S_n) \subseteq P_{n+1}$ is the set of elements that can modify the causal order of P_n, and its causal presupposition set $E_{\mathrm{pre}} \subseteq P_n$ is the minimal causal condition required for the execution of S_n. When there exists a map T_n such that $T_n(E_{\mathrm{pre}}) \subseteq \mathrm{Scope}(S_n)$, S_n modifies the presuppositions of its own execution, generating a closed causal chain of the form $a \to T_n(a) \to \cdots \to a$, which directly destroys the antisymmetry of the partial order and causes a global collapse of the transitivity of the causal order.

The critical level N_c is defined as the smallest n such that $\dim(P_n) > n$. Above N_c, the number of independent order directions required for objectification exceeds the level index, and any self-referential map inevitably triggers circular dependence.

B.4 Comparison between Logical Critical Point and Thermodynamic Capping

The main text, based on the dimension growth law and the thermodynamic cost law, has asserted $N_c \geq n_{\max}$, i.e., the causal paradox cutoff lies above the thermodynamic bankruptcy point. A supplementary argument is given here.

From the dimension growth law, $\dim(P_n) \geq \dim(P_1) + (n-1)$. Taking $\dim(P_1) \geq 1$, we have $\dim(P_n) \geq n$. The condition for the strict inequality $\dim(P_n) > n$ to hold is $\dim(P_n) \geq n+1$, i.e., there exist at least $n+1$ independent order directions in the system.

Each independent order direction requires at least one bit in the meta-description to characterize its preference orientation. Representing $d_n = \dim(P_n)$ independent order directions requires at least d_n bits of meta-rule information. Under the physical encoding scheme, adding the representation of one independent order direction corresponds to at least one bit of discarding-type rule replacement, whose energy cost is no less than $k_B T \ln 2$. Let the maximum number of independent order directions that the system's total energy E_{total} can support be $d_{\max}$; then $d_{\max} \cdot (k_B T \ln 2) \le E_{\text{total}}$, i.e., $d_{\max} = n_{\max}$. By the thermodynamic capping theorem, $n_{\max} = \lfloor E_{\text{total}}/(k_B T \ln 2) \rfloor$ is the maximum level the system can reach.

Therefore, to satisfy $\dim(P_n) > n$, i.e., $\dim(P_n) \ge n+1$, we must have $n+1 > n_{\max}$ (otherwise the cumulative energy cost of the required $n+1$ independent order directions would already exceed E_{total}). From this we obtain $N_c \ge n_{\max} + 1$. The logical critical point lies strictly above the thermodynamic bankruptcy point, with at least one full level separating them.

This conclusion guarantees that the physical system cannot actually reach the causal paradox level before energy exhaustion; the paradox cutoff purely constitutes a logical safety redundancy at a higher position, thereby together with the thermodynamic capping locking the unique optimal time level.

Appendix C: Responses to Potential Counterexamples and Objections

C1 Open Systems and Environmental Heat Baths

Objection: If the system continuously draws free energy from an external heat bath, E_{total} can be regarded as infinite, then $n_{\max}$ is infinite and time upgrade can proceed forever.

Response: The system referred to in this paper is a locally definable physical system. If the system and environment are considered together as a whole, the total energy of this composite system is finite (the cosmological constant and the total energy of the observable universe are bounded). The maximum upgrade level is determined by the total energy of the composite system and remains finite. Even if one introduces an infinite-temperature heat bath in a thought experiment, or assumes a theoretically infinite energy source, Landauer's principle still requires each erasure to consume $k_B T_c \ln 2$, where T_c is the environmental temperature at which the computation is performed; under infinite energy, T_c cannot simultaneously remain finite without diverging, and once the heat bath temperature is infinite, the cost of erasing one bit also diverges, making continued upgrades impossible.

If the system continuously absorbs low-entropy energy from the environment while expelling high-entropy waste heat (like living systems or non-equilibrium steady-state systems), it may locally appear to sustain long-range upgrade capability without violating the second law of thermodynamics. In this regard, it is necessary to clarify the analytical boundary of this paper: the cumulative cost of a complete sequence of time upgrades within a single system is borne by the energy budget of that system. If the system remains open and continuously upgrades by net absorption of free energy, the "bill" for the upgrade is ultimately paid by the composite formed by the environment and the system together. Extending the analytical boundary to this composite, the theorem applies equally: the composite total energy is finite, and the composite $n_{\max}$ is finite. For a single system that remains open, E_{total} should be understood as "the maximum energy budget that the system can call upon from within its causal boundary during the upgrade period" (including pre-appropriable environmental free energy), and the structure of the theorem remains unchanged. Open systems do not constitute counterexamples; they merely require correct delineation of the analytical boundary.

C2 Quantum Vacuum Fluctuations and Negative Energy

Objection: Can one bypass energy consumption by exploiting the Casimir effect or quantum tunneling to obtain "negative energy" or borrow energy from the vacuum?

Response: The Landauer lower bound applies to the entropy production caused by information erasure in a thermal equilibrium environment; it does not prescribe the form of the energy source, but only stipulates the minimum heat dissipated into an environment at temperature T. Quantum fluctuations are subject to constraints such as the averaged weak energy condition (AWEC) and cannot stably and continuously provide usable negative energy to offset the necessary dissipation of each upgrade. Even in strong quantum gravitational spacetime foam, such effects at most influence microscopic-scale fluctuations and cannot systematically cancel the information-energy cost demanded by the second law of thermodynamics. Therefore, the energy accounting law remains valid.

C3 Avoiding Erasure Energy Cost by Modifying Causal Rules

Objection: Could one modify causal rules at a higher time level so that "information erasure" no longer consumes energy, thereby circumventing thermodynamic capping?

Response: Such an operation constitutes a self-referential modification. Before reaching N_c (if one could reach it), it is itself a time upgrade, which necessarily entails objectifying and encoding the old rules, thus consuming at least one bit of erasure energy. If the new rule abolishes the physical law of energy consumption, it means the modified lower-level system would lose predictable thermodynamic behavior, and the causal order would consequently lose its transitivity—which is precisely the characteristic of the causal degradation phase. However, before that, executing this modification already requires paying the energy cost within $n_{\max}$, and due to energy bankruptcy, the system cannot actually reach the level at which such self-referential sabotage could be completed. Thermodynamic capping stands in front, the logical trap behind; the two constitute a double insurance.

C4 Systems with Extremely Large Energy (such as the Universe as a Whole)

Objection: If the entire observable universe is taken as the system, E_{total} is extremely large and $n_{\max}$ is correspondingly very large; could time upgrades reach extremely high levels?

Response: This is indeed possible, but even so, $n_{\max}$ remains a finite integer, and the fixed point still exists uniquely. This scenario precisely illustrates the universality of the theorem: no matter how large the system is, as long as it is finite, the maximum level is finite; and standard thermodynamics and causal logic do not break down at the cosmic scale. If future cosmological evidence indicates that the total energy of the universe is arbitrarily large or infinite, then the boundary conditions of the theorem would need to be re-examined; but within the current framework of physics, finite energy is a reasonable premise for the argument.

Appendix D: Fixed Point under Cosmic Scales and Extreme Gravity—Rigorous Derivation of the Existence Margin

This appendix extends the thermodynamic capping theorem from weakly self-gravitating local systems to strong gravity and the cosmic scale as a whole. The core conclusion is that, under the above extreme conditions, the termination of the time upgrade may be dominated by the **existence margin** (rather than the energy margin), but the existence and uniqueness of the fixed point remain unchanged. The derivation does not introduce any external speculative framework and is based solely on the partial order dimension and information bound tools already established in the main text.

D.1 Information Bound Correction in the Strong Gravity Regime

The main text adopts the Bekenstein bound $I_{\max}^{\mathrm{Bek}} = 2\pi ER/(\hbar c \ln 2)$ as the information budget. This bound holds under weak self-gravity. When the system's gravity strengthens and R approaches its Schwarzschild radius $R_s = 2GE/c^4$, the holographic bound

$$I_{\max}^{\mathrm{holo}} = \frac{A}{4l_P^2 \ln 2} = \frac{\pi R^2}{l_P^2 \ln 2}$$

tightens into a stricter constraint ('t Hooft 1993; Susskind 1995), where $l_P = \sqrt{\hbar G/c^3}$ is the Planck length. The actual information capacity of the system is

$$I_{\max} = \min\left(I_{\max}^{\mathrm{Bek}},\ I_{\max}^{\mathrm{holo}}\right).$$

At macroscopic and even cosmological scales, the holographic bound almost inevitably tightens before the Bekenstein bound. Substituting R as the characteristic scale of the system, the information budget is no longer determined by the bulk energy but locked by the boundary area.

D.2 Information Cost of Partial Order Dimension and the Existence Margin

Appendix B.2 of the main text has strictly proved that each objectification upgrade necessarily introduces at least one new independent order direction, i.e., $\dim(P_{n+1}) \geq \dim(P_n) + 1$. Taking the initial value $\dim(P_1) \geq 1$, we have

$$\dim(P_n) \geq n.$$

Each independent order direction requires at least one physical bit for its representation in the meta-description. Therefore, a **necessary condition** for the causal structure of level n to be physically realizable is that the system possesses at least n bits of available information capacity:

$$n \le I_{\max}.$$

In the weak self-gravity regime, $I_{\max} \approx I_{\max}^{\mathrm{Bek}}$ usually far exceeds $n_{\max}$ (determined by the energy bound), so energy bankruptcy precedes information saturation, and $n_{\mathrm{actual}} = n_{\max} = \lfloor E_{\mathrm{total}}/(k_B T \ln 2) \rfloor$.

However, in the strong gravity or whole-universe regime, $I_{\max}$ may be determined by the holographic bound $I_{\max}^{\mathrm{holo}}$, and its value may be **smaller** than $n_{\max}$. In this case, even if the system possesses sufficient energy to pay the thermodynamic cost of $n_{\max}$ erasures, it does not have enough information capacity to **carry** the $n_{\max}$ independent order directions necessary for the causal structure of the $n_{\max}$-th level.

Hence, we define the **existence margin**:

$$n_{\mathrm{exist}} = \lfloor I_{\max} \rfloor.$$

The physically realizable highest level of the system is

$$n_{\mathrm{actual}} = \min(n_{\max},\ n_{\mathrm{exist}}).$$

When $n_{\mathrm{exist}} < n_{\max}$, the time upgrade is forced to terminate even before the energy budget is exhausted, because no more independent order directions can be represented. This is capping dominated by the existence margin.

D.3 Explicit Expression of the Existence Margin under the Holographic Bound

Under the condition that the holographic bound is the tight constraint,

$$n_{\mathrm{exist}}^{\mathrm{holo}} = \left\lfloor \frac{\pi R^2}{l_P^2 \ln 2} \right\rfloor.$$

Taking the observable universe as an example: $R \approx 4.4 \times 10^{26}$ m, $l_P \approx 1.6 \times 10^{-35}$ m, substituting yields $n_{\mathrm{exist}}^{\mathrm{holo}} \approx 10^{122}$. This is the maximum number of independent causal order directions that the holographic screen of the observable universe can encode. Meanwhile, the energy account corresponding to $n_{\max}$, given the total cosmic energy $E_{\mathrm{total}} \approx 10^{70}$ J and the CMB temperature $T \approx 2.7\,\mathrm{K}$, is of order 10^{90}. Comparing these two numbers:

$$n_{\max} \approx 10^{90}, \quad n_{\mathrm{exist}}^{\mathrm{holo}} \approx 10^{122}.$$

Under these parameters, $n_{\max} < n_{\mathrm{exist}}$; **the energy margin still arrives before the existence margin**. This is not a failure of the theorem; on the contrary, it shows that the existence margin is a second boundary that is looser than the thermodynamic capping but logically independent. The fixed point is uniquely determined by the stricter of the two:

$$n_{\mathrm{actual}} = n_{\max} \approx 10^{90}.$$

D.4 Dimensional Fragmentation and Nonlinear Behavior of Partial Order Dimension

The above analysis assumes that $\dim(P_n)$ grows strictly linearly with n, making $n \leq I_{\max}$ constitute the existence margin. However, as the holographic bound limit is approached, the partial order dimension may exhibit a **nonlinear saturation** phenomenon: when the system's information capacity approaches saturation, the representation bits required for newly added independent order directions cannot be continuously allocated, causing $\dim(P_n)$ to stop growing or be forcibly folded at some n.

This phenomenon can be derived independently from the closed causal chain logic of the main text's Section B.3. Suppose the system has exhausted its $I_{\max}$ bits of information capacity to represent $\dim(P_n) = n$ independent order directions. If the $(n+1)$-th upgrade is now attempted, a new independent order direction must be created, but no usable physical bits remain—forcing the coder/decoder to **map the new direction onto one of the already existing representation directions**, thereby forcibly identifying two independent order directions. This is precisely the **folding operation** described in Section B.3 of the main text, directly producing a closed causal chain and destroying the transitivity of the partial order.

Thus, even if one could hypothetically continue upgrading by "overdrawing" energy in some way, the folding effect at the existence margin n_{exist} would inevitably trigger causal degradation. n_{exist} is not an asymptotic line that can be approached indefinitely, but a hard cutoff where the causal structure self-destructs.

D.5 Uniqueness of the Fixed Point (Cosmic Scale)

With the introduction of the existence margin, the actual highest level is $n_{\text{actual}} = \min(n_{\max}, n_{\text{exist}})$. For any given boundary conditions (E_{total}, R, T), both $n_{\max}$ and n_{exist} are uniquely determined integers, and their minimum is unique. Therefore:

- If $n_{\max} \leq n_{\text{exist}}$, capping is dominated by thermodynamic energy bankruptcy, and the fixed point is at $n_{\max}$;
- If $n_{\text{exist}} < n_{\max}$, capping is dominated by information capacity saturation, and the fixed point is at n_{exist}.

In either case, the fixed point exists uniquely. The core conclusions of Theorem 1 and Theorem 2—that time upgrades must terminate after finite steps and that the termination level is unique—continue to hold on the strong-gravity and cosmic scales.

D.6 Connection to the "Failed Universe" Intuition (Only a Note)

The derivation of this appendix provides a physical criterion, independent of external speculation, for distinguishing "successful universes" from "failed universes": **a successful universe is one for which both $n_{\max}$ and n_{exist} are finite positive integers, and whose internal causal structure can achieve self-consistent closure within n_{actual} levels; a failed universe is a parameter configuration for which $I_{\max}$ is insufficient to support any nontrivial partial order structure (i.e., $n_{\text{exist}} < 1$).** Under this definition, the observable universe belongs to the class of successful universes.

As for other parameter regions in the multiverse, their existence margins can be rigorously computed from the correspondence between the holographic bound and the partial order dimension, without introducing an additional "self-referential depth" parameter.

Summary: The existence margin, corrected by the holographic bound, is the logical dual of thermodynamic capping—the former restricts "how many erasures you can afford," the latter restricts "how many independent causal directions you can represent." The minimum of the two locks the unique fixed point. The derivation of this appendix strictly relies on the tools of the main text and introduces no speculative assumptions, endowing the universality of the time rigidity theorem on cosmic scales with the same mathematical reliability as in the local case.